$f: M^2 \to N^2$ is isometry iff $f^*(G_n) = G_m \Leftrightarrow G_M(X,Y) = G_N(df(x), df(Y))$, G_M, G_N Rieman. metric on M, N.
$\gamma: [a,b] \to \mathbb{R}^n$ regular if $\forall t \in [a,b]: \gamma'(t) \neq 0$ where $\gamma(t) = (x^1(t), \ldots, x^n(t)) \Rightarrow \gamma'(t) = (\dot{x}^1(t), \ldots, \dot{x}^k(t))$.
$\mathcal{D}_t$ system of equations passing through point (x_0, y_0), direction vector $(u(t), v(t))$. Get $w(t)$ by substitution (x_0, y_0). Then we have system of equations:

$$\begin{cases} u(t)x(t) + v(t)y(t) + w(t) = 0 \\ u'(t)x(t) + v'(t)y(t) + w'(t) = 0 \end{cases} \Leftrightarrow \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -w(t) \\ -w'(t) \end{pmatrix} \Leftrightarrow AX = W$$

Envelope given by $X = A^{-1}W$ iff A invertible.

γ	$\gamma(t)$	$\gamma(s,t)$	
Normal	$n(t) = \frac{\gamma"(t)}{\|\gamma"(t)\|}$	$n(s,t) = \frac{\frac{\partial}{\partial s}\gamma(s,t) \times \frac{\partial}{\partial t}\gamma(s,t)}{\|\frac{\partial}{\partial s}\gamma(s,t) \times \frac{\partial}{\partial t}\gamma(s,t)\|}$	
Curvature	$k(s) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}$ $(x(t), y(t), c) \Rightarrow k(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y})^2)^3}$	$k(s,t) = (-1)^{n-1} \frac{\det(Q(s,t))}{\det(G(s,t))}$	$k(p) = \det(d_p n)$ $k = 0$: eucl. plane $k = 1 : \mathbb{S}^2$ $k = -1 : \mathbb{H}$

$\mathbb{H} = \{u + iv = z \in \mathbb{C} | \text{Im}(z) > 0\}$, hyperbolic plane.
Planar curve: Curve entirely contained in plane.
Flat metric: Metric with curvature equal to zero at every point.
1st fund. form: $G = \begin{pmatrix} \langle\gamma_s,\gamma_s\rangle & \langle\gamma_s,\gamma_t\rangle \\ \langle\gamma_t,\gamma_s\rangle & \langle\gamma_t,\gamma_t\rangle \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ 2nd fund. form: $Q = \begin{pmatrix} \langle\gamma_{ss},n\rangle & \langle\gamma_{st},n\rangle \\ \langle\gamma_{ts},n\rangle & =\langle\gamma_{ts},n\rangle \end{pmatrix}$
p is 1) elliptic: $\det(Q) > 0$, 2) hyperbolic: $\det(Q) < 0$, 3) parabolic: $\det(Q) = 0 \& Q \neq 0$.
Length curve $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ on $I = [a,b] \Rightarrow \text{len}(\alpha) = \int_a^b \sqrt{\sum_{i,j=1}^{2} g_{ij}(\alpha(t))\dot{\alpha_i}(t)\dot{\alpha_j}(t)dt}$

$f(s)$ param. curve γ by arc length $\Rightarrow f_a(s) = f(s) + an(s)$ param. of parallel curves γ_a.
Huygens principle: evolute $= \{a | a$ singul. point of $f_a(s)\} \Leftrightarrow \{a | \rho(s) = a\} \Leftrightarrow \left\{a \middle| \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} + \frac{1}{k(s)} n(s) = a\right\}$
$M^{n-1} \hookrightarrow \mathbb{R}^n$ reg. surface or subset $\mathbb{R}^n$ with riemannian metric $(g_{ij}(p))$. Then geodesic on M^{n-1} smooth curve $\gamma: I \to M^{n-1}$ of constant speed, locally realizing:

$$d(p,q) := \inf_\gamma \int_{t_1}^{t_2} \sqrt{g_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)dt} \quad \gamma \text{ smooth curves on } M^{n-1} \text{ s.t. } \gamma(t_1) = p, \gamma(t_2) = q$$

M^{n-1} param. by $(u^1, \ldots, u^{n-1})$. If $\gamma: I \to M^{n-1}$ geodesic, it satisfies:

$$\ddot{u}^i + \Gamma^i_{jk}\dot{u}^j\dot{u}^k = 0$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{is}}{\partial u^k} + \frac{\partial g_{ks}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^s} \right) \text{ Christoffel symbols, sum over s=1 till s=n-1}$$

$$M^{n-1} \text{ regular } \Rightarrow \Gamma^i_{jk} = \langle r^i, r_{jk} \rangle \text{ with } \langle r^i, r_j \rangle = \delta^i_j$$

$$\text{geo. equa. } \frac{d^2\gamma(t)}{dt^2} \parallel n(\gamma(t)), \text{with } n \text{ normal to } M^{n-1}$$

Geodesics $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $\mathbb{H}$ if $\begin{cases} \ddot{\gamma_1} - 2\gamma_2^{-1}\dot{\gamma_1}\dot{\gamma_2} = 0 \\ \ddot{\gamma_2} - \gamma_2^{-1}(\dot{\gamma_2}^2 - \dot{\gamma_1}^2) = 0 \end{cases}$.

Local Gauss-Bonnet theorem:
M^2 regular surface, rieman. metric $G(p)$ and $k = k(p) : M^2 \to \mathbb{R}$ gauss. curv. Geodesic triangle $ABC \subset M^2$ with interior angles $\alpha, \beta, \gamma \Rightarrow \int k ds = \alpha + \beta + \gamma - \pi$ with $ds = \sqrt{\det(G)} du^1 \wedge du^2$ area element of M^2.

E vector space other set $\mathcal{E}$. Map $f: \mathcal{E} \times \mathcal{E} \to E$, $f(A,B) = \vec{AB}$ s.t.

1. $\forall A \in \mathcal{E}, f(A, \bullet): \mathcal{E} \to E$ is a bijection.
2. $\forall A, B, C \in \mathcal{E}\ \vec{AB} + \vec{BC} = \vec{AC}$

then $\mathcal{E}$ affine space.
Parallelogram rule: $AB = CD \Leftrightarrow BD = AC$. Addition: $A \in \mathcal{E}, u \in E$ let B s.t. $\vec{AB} = u \Rightarrow B := A + \vec{AB}$.
$\mathcal{E}$ affine space, $\mathcal{F} \subset \mathcal{E}$ if $\forall A \in \mathcal{F}$ s.t. $F = \{\vec{AB}, B \in \mathcal{F}\}$ vector subspace E. $\mathcal{F}$ is directed by F.

Let $\mathcal{E}$ be an affine space directed by E and $\mathcal{F}$ an affine subspace directed by F. $\dim(\mathcal{F}) = \dim(F)$.
$A \in \mathcal{E}, F \subset E \Rightarrow \exists! \mathcal{F} \subset \mathcal{E}$ s.t. $A \in \mathcal{F}$.
$\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{E}$ satisfy $\mathcal{D}_1 \parallel \mathcal{D}_2$ iff same direction D.$\mathcal{F}_1 \parallel \mathcal{F}_2$ iff $\mathcal{F}_i = A_i + F$ for $i = 1, 2$.
$\forall \mathcal{D} \in \mathcal{E}$ s.t. $A \notin D, \exists! \mathcal{D}'$ s.t. $A \in \mathcal{D}'$ and $\mathcal{D}' \parallel \mathcal{D}$.
$f : \mathcal{E} \to \mathcal{F}$ affine if $\exists A \in \mathcal{E}$ s.t. $\vec{AB} = f(\vec{AB}) = f(A)\vec{f}(B)$ is lin. mapping E to F.
If E, F bases $\{e_i\}, \{f_i\}$ then $\forall f : \mathcal{E} \to \mathcal{F}$ affine, has form $f(x) = Ax + b$ with $A : E \to F, b \in \mathcal{F} \simeq F$.
Group invertible affine maps $n-$dim.affine space $\mathcal{E}$,called $\mathrm{Aut}(\mathcal{E}) \cong GL(n, \mathbb{K}) \ltimes \mathbb{K}^n$.
Thales' theorem $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{E}$ and $d \parallel d' \parallel d" \Rightarrow \frac{A_1\vec{A_1}"}{A_1\vec{A}'_1} = \frac{A_2\vec{A_2}"}{A_2\vec{A}'_2} = \lambda$. Conv. $B \in \mathcal{D}_1$ s.t. $\frac{A_1\vec{B}}{A_1\vec{A}'_1} = \frac{A_2\vec{A_2}"}{A_2\vec{A}'_2} \Rightarrow B = A_1"$.

$\dim(\mathcal{E}), \dim(\mathcal{F}) \geq 2$. $\varphi : \mathcal{E} \to \mathcal{F}$ bijec. s.t. $\forall A, B, C \in \mathcal{E} \Rightarrow \varphi(A), \varphi(B), \varphi(C)$ collin. $\Rightarrow \varphi$ isom. aff. mapping.

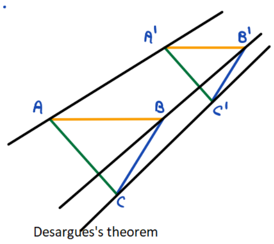
$A_0, \ldots, A_k \in \mathcal{E}$ are affinely independent if $\mathcal{F} = \mathrm{span}\{A_0, \ldots, A_k\}$ has dim. k. ($\{A_i\}$ is AI)
$\mathrm{span}\{A_0, \ldots, A_n\} = \mathcal{E}$ and $\{A_i\}$ is AI $\Rightarrow \{A_i\}$ form affine frame for $\mathcal{E}$.
$\{A_i\} \in \mathcal{E}$ AI, k+1 weights $\alpha_0, \ldots, \alpha_k \in \mathbb{K}$ s.t. $\sum \alpha_i \neq 0$.Barycenter/center of mass of weighted points (A_i, α_i) is point $G := O + \frac{\sum_{i=0}^{k} \lambda_i O\vec{A}_i}{\sum_{i=0}^{k} \lambda_i}$ with $O \in \mathcal{E}$ arbitrary. Note: Barycenter is unique G s.t. $\frac{1}{\lambda} \sum_{i=0}^{k} \lambda_i G\vec{A}_i = 0$.

$\forall C \in \mathrm{span}\{A_0, \ldots, A_k\}, \exists \lambda_0, \ldots \lambda_k \in \mathbb{K}, \sum_{i=0}^{k} \lambda_i \neq 0$ s.t. C barycenter of (A_i, λ_i).
Observe $(\lambda_0, \ldots, \lambda_k) \& (\overline{\lambda_0}, \ldots, \overline{\lambda_k})$ same barycenter iff $(\overline{\lambda_0}, \ldots, \overline{\lambda_k}) = (\beta\lambda_0, \ldots, \beta\lambda_k)$ for $\beta \neq 0$.
$(\lambda_0 : \ldots : \lambda_k)$ are BARYCENTRIC/HOMOGENEOUS COORDINATES of G in affine frame $A_0, \ldots, A_k$.
Pappus thm:$\mathcal{D}, \mathcal{D}' \in \mathcal{E}$. If $A, B, C \in \mathcal{D} \& A', B', C' \in \mathcal{D}'.AB' \parallel BA', BC' \parallel CB' \Rightarrow AC' \parallel CA'$.
Desargues's theorem $ABC, A'B'C'$ two triangles without common vertex in affine plane. If $AB \parallel A'B', AC \parallel A'C'$ and $BC \parallel B'C'$. Then lines AA', BB', CC' are parallel or intersection in single point.

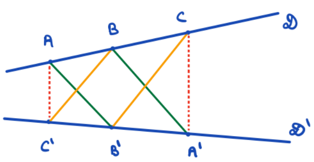
$f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $\langle f(u) - f(v), f(u) - f(v) \rangle = \langle u - v, u - v \rangle \Rightarrow f(x) = Ax + b$, with A orthog:$AA^T = A^T A = E$.
reflection s_e in hyperplane e in $\mathbb{R}^n$ def. by $s_e(u) = 2\mathrm{Pr}_e(u) - u$ with $\mathrm{pr}_e(u)$ is the orthogonal projection of u on e. Orientation reserving isometries on $\mathbb{R}^n$. Every isometry of $\mathbb{R}^n$ is a composition of reflections.
Inversion,$I(O, R^2)$ in a sphere $S^{n-1}_R(O) \hookrightarrow \mathbb{R}^n$ of radius R def. by $I(O, R)(A) = O + \frac{R^2 O\vec{A}}{\|O\vec{A}\|^2}$
Inversions in circles: $I(O, R) \circ I(O, R) = \mathrm{id}$.
Inversion in circle $\mathbb{R}^2 \cong \mathbb{C}$ with center on real axis,$y = 0$. Inversion is isometry on $\mathbb{H}$ with hyperbolic metric $\frac{dx^2+dy^2}{y^2}$.
$P_n(\mathbb{K}) = \mathbb{K}^n \cup P_{n-1}(\mathbb{K}) = \mathbb{K}^n \cup \mathbb{K}^{n-1} \cup \ldots \cup \mathbb{K} \cup \{\mathrm{pt}\}$. Therefore $P_n(\mathbb{K})$ is projective completion of affine space $\mathbb{K}^n$ by adding hyperplane $P(\{\sum_{i=0}^{n} x^i = 0\})$ at infinity.

$PSL(n, \mathbb{K}) = \{f : P_{n-1}(\mathbb{K}) \to P_{n-1}(\mathbb{K}) | f \text{ bijective}\} \cong SL(n, \mathbb{K})/\{\pm E\}$

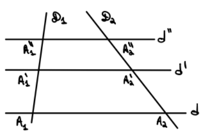
$\mathrm{PSL}(2, \mathbb{C})$ generated by even number inversions and reflections in arbitrary circles or lines in $\mathbb{C}$.
$\mathrm{PSL}(2, \mathbb{R})$ group orientation preserving isom. of $\mathbb{H}$,generated by even number inversion and reflections in circles or lines orthogonal to $\{\mathrm{Im}(z) = 0\}$.
E vector space over $\mathbb{K}$, consider $P(E \bigoplus \mathbb{K})$ projective space and $f : P(E \bigoplus \mathbb{K}) \to P(E \bigoplus \mathbb{K})$ projective transformation preserving hyperplane at infinity. Then induced map on $\mathcal{E} = \{(u, \lambda) \in E \bigoplus \mathbb{K} | \lambda = 1\}$ is affine (so $f : \mathcal{E} \to \mathcal{E}$ affine).
Conv. every affine bijection $g : \mathcal{E} \to \mathcal{E}$ can be extended to projective transf. $g : P(E \bigoplus \mathbb{K}) \to P(E \bigoplus \mathbb{K})$.
V vector space then dual vector space $V^* = \{\alpha | \alpha : V \to K \text{ linear}\}$.
Observe that $(\mathbb{K}^{n+1})^* \cong \mathbb{K}^{n+1}$. Not canocially, so depends on basis.
We see that $d : (k -$ dim. vector subspace $\mathbb{K}^{n+1}) \to ((n + 1 - k) -$ dim vector subspace of $\mathbb{K}^{n+1})$,i.e.
projective duality



Desargues's theorem



Pappus' theorem


Thales' theorem